Phys 410
Fall 2015
Lecture #9 Summary
29 September, 2015

We considered the *driven* damped harmonic oscillator and resonance in detail. The response to a cosine ($\cos(\omega t)$) driving force in the long-term limit is: $x(t) = A\cos(\omega t - \delta)$, where ω is the frequency of the driving force. This represents the long-time persistent solution of the motion. It shows that the oscillator eventually adopts the same frequency as the driving force.

The equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$ involves a linear operator $L = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$ acting on the displacement function $x(t)$ and relating it to the driving force $f(t)$ as $L\,x(t) = f(t)$. The linearity property means that the operator can operate on any number of solutions at the same time: $L\,[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 f_1(t) + \alpha_2 f_2(t)$, for arbitrary weighting coefficients α_1 and α_2. In other words, when two driving forces act we can consider the response of the system to each force individually, and then simply add the two solutions to get the full response of the oscillator. (Note that this will NOT work if we had a non-linear operator, for example in the case of a damping force quadratic in the velocity, leading to a term $\gamma\dot{x}^2$.) This property allows us to consider an arbitrary periodic driving force $f(t+T) = f(t)$, where T is the period of the driving force, as being made up of an infinite superposition of cosine driving forces: $f(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t)$, where we assume that a Fourier cosine expansion is adequate to describe the periodic driving force. The linearity of the problem allows us to write down the general solution as $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$, with $A_n = f_n/\sqrt{(\omega_0^2 - (n\omega)^2)^2 + 4\beta^2 n^2\omega^2}$ and $\delta_n = \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - (n\omega)^2}\right)$. With this we can describe the motion of the driven system subjected to more general periodic driving forces, such as a triangle wave, periodic pulsed driving forces, etc.

We can go one step further and consider the solution for an arbitrary non-periodic driving force $F(t)$. In this case we consider the driving force to be a series of sharp impulses applied to the initially stationary oscillator at its equilibrium position. The force $F(t)$ is approximated as a series of impulses at periodic time steps (separated by a short time τ), and the solutions for the position of the oscillator after each impulse applied at time step t_n is given by $x(t) = \begin{cases} 0 \text{ for } t < t_n \\ \frac{v_0}{\omega_1} e^{-\beta(t-t_n)} \sin[\omega_1(t-t_n)] \text{ for } t \geq t_n \end{cases}$. Here $v_0$is the magnitude of the initial kick of the oscillator from rest. Note that each solution is independent of all the others, and each assumes an initial condition of $x_n(t = t_n) = 0$. The driving force can be written as a sum of many individual impulses as $f(t) = F(t)/m = \sum_{n=-\infty}^{N} f_n(t_n)$, stretching back to $t = -\infty$ up to the present time step $t_N = t$. Making use of the linearity of the L operator discussed above, we can form the solution to this arbitrary driving force as a sum of all of the elementary responses to all of the previous impulses as follows: $x(t) = \sum_{n=-\infty}^{N} \frac{f_n(t_n)\tau}{\omega_1} e^{-\beta(t-t_n)} \sin\big(\omega_1(t-t_n)\big)$, valid for $t > t_N$. As time goes forward, you have to add more impulses to the sum (assuming that the driving function $f_n(t_n)$ is non-zero). We recover the continuous forcing function by taking the time interval $\tau \to dt'$ and $t_n \to t'$ and letting the sum go over into an integral: $x(t) = \int_{-\infty}^{t} \frac{f(t')}{\omega_1} e^{-\beta(t-t')} \sin\big(\omega_1(t-t')\big)\, dt'$. This can be written as $x(t) = \int_{-\infty}^{t} F(t')\, G(t,t') dt'$, defining the Green's function $G(t,t') = \begin{cases} \frac{e^{-\beta(t-t')} \sin\left(\omega_1(t-t')\right)}{m\omega_1} \text{ for } t \geq t' \\ 0 \text{ for } t < t' \end{cases}$. Note that this Green's function assumes that the particle starts from a state of rest at the equilibrium point. One has to re-derive the Green's function for a different initial condition.

We did an example of the Green's function solution for a non-periodic driving force that acts for a more-or-less fixed time interval, of the form: $F(t) = \begin{cases} 0 \text{ for } t < 0 \\ F_0 e^{-\gamma t} \text{ for } t \geq 0 \end{cases}$. The solution for the response of the mass is $x(t) = \begin{cases} 0 \text{ for } t < 0 \\ \frac{F_0/m}{(\gamma-\beta)^2+\omega_1{}^2}\left[e^{-\gamma t} - e^{-\beta t}\left(\cos(\omega_1 t) - \frac{\gamma-\beta}{\omega_1}\sin(\omega_1 t)\right)\right] \text{ for } t \geq 0 \end{cases}$. This complicated function describes the motion of the harmonic oscillator while the force is acting, and for all times into the future.

We moved on to the question of how to make Newton's Laws of motion work in non-inertial reference frames. This turns out to be useful for a number of reasons. First we often insist on using coordinate systems that are non-inertial, such as the (Latitude, Longitude, Altitude) "GPS" reference frame attached to the surface of the rotating earth. Secondly, some physical problems are easier to attack when seen from non-inertial reference frames, such as the "co-rotating frame" rotating at the Larmor precession frequency in NMR. Another example is the description of small oscillations about an equilibrium point in a non-inertial reference frame.

We considered first the case of a reference frame undergoing constant linear acceleration $\vec{A}$. By comparing a description of the motion of an object as seen from an inertial reference frame to that same object seen from a non-inertial reference frame, we concluded that Newton's second law in the non-inertial reference frame must be written as $m\ddot{\vec{r}} = \vec{F}_{net} - m\vec{A}$. The "inertial force" $\vec{F}_{inertial} = -m\vec{A}$ must be added to the net force to make the equation of motion work in the non-inertial frame. We experience this inertial force as a backwards force when sitting in an aircraft that is accelerating down the runway for takeoff.

Making Newton's second law work in a rotating reference frame is more of a challenge. Consider a rigid body moving through space. A rigid body is one in which the distances between the particles do not change during the motion. We can start by describing the motion of the center of mass $\vec{R}_{CM}(t)$ and treat it as the motion of a particle of mass M equal to the total mass of the object. With an extended rigid body we have the additional degree of freedom that the object can also be rotating or tumbling. We can treat the center of mass as a stationary point during the motion. Euler's theorem says that the most general motion of that object is a rotation about an axis going through the center of mass. This rotational motion is specified by a direction of the rotation axis and the magnitude of the rotation rate. Rotation is specified by an axis of rotation $\hat{u}$, and a rate ω, so that $\vec{\omega} = \omega\hat{u}$. The rotation axis goes through the fixed point in the object. We found that the linear velocity of a particle at location $\vec{r}$ inside or on the object is given by $\vec{v} = \vec{\omega} \times \vec{r}$. In other words $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, or in general for any vector $\vec{e}$ in the rigid body $\frac{d\vec{e}}{dt} = \vec{\omega} \times \vec{e}$.